Contents

Chapter 4

Laplace transform

4.1 Definition of the Laplace transform

Integral transform: Given $K(s,t)$ let

$$F(s) = \int_0^\infty K(s,t)f(t)dt.$$

Definition 4.1.1. $f(t)$ is given for $t \geq 0$. The improper integral

$$\boxed{F(s) = \int_0^\infty e^{-st}f(t)dt} \tag{4.1}$$

is called the **Laplace transform** of f(provided the integral exists) and denoted by $\mathcal{L}\{f(t)\}$. In this case f is the inverse of $F(s)$ and write

$$f(t) = \mathcal{L}^{-1}\{F(s)\}.$$

Example 4.1.2. Find the Laplace transform of $f(t) = 1$.

$$\boxed{\mathcal{L}\{1\} = \int_0^\infty e^{-st}dt = \frac{1}{s},\ s > 0}$$

$$\boxed{\mathcal{L}\{t\} = \int_0^\infty te^{-st}dt = \frac{1}{s^2},\ s > 0}$$

$$\boxed{\mathcal{L}\{t^n\} = \int_0^\infty t^n e^{-st}dt = \frac{n!}{s^{n+1}},\ s > 0}$$

Example 4.1.3. Find the Laplace transform of $f(t) = e^{ct}$. We have for $s > c$

$$\mathcal{L}\{e^{ct}\} = \int_0^\infty e^{-st} e^{ct} dt = \frac{1}{c-s} e^{(c-s)t} \Big|_0^\infty = \frac{1}{s-c}, \quad s > c$$

Example 4.1.4. Find the Laplace transform of $f(t) = \cos at$.

We have for $s > 0$

$$\begin{aligned} F(s) &= \mathcal{L}\{\cos at\} = \int_0^\infty e^{-st} \cos at \, dt \\ &= \frac{e^{-st} \sin at}{a} \Big|_0^\infty + \frac{s}{a} \int_0^\infty e^{-st} \sin at dt = \frac{s}{a} \int_0^\infty e^{-st} \sin at \, dt \\ &= \frac{s}{a} \left(- \frac{e^{-st} \cos at}{a} \Big|_0^\infty - \frac{s}{a} \int_0^\infty e^{-st} \cos at \, dt \right) \\ &= \frac{s}{a} \left(\frac{1}{a} - \frac{s}{a} F(s) \right). \end{aligned}$$

Solving for $F(s)$ we have

$$F(s) = \mathcal{L}\{\cos at\} = \frac{s}{s^2 + a^2}, \quad s > 0$$

Example 4.1.5. The Laplace transform of trig. functions is easily deduced from those of exponential functions. Noting that $e^{iat} = \cos at + i \sin at$, we obtain

$$\frac{1}{s - ia} = \mathcal{L}\{e^{iat}\} = \mathcal{L}\{\cos at + i \sin at\} = \frac{s + ia}{s^2 + a^2}.$$

Comparing real and imaginary parts

$$\mathcal{L}\{\cos at\} = \frac{s}{s^2 + a^2}, \quad \mathcal{L}\{\sin at\} = \frac{a}{s^2 + a^2}$$

Sufficient condition for the existence of Laplace transform

Theorem 4.1.6. *Suppose f is* ***piecewise continuous*** *on $[0, \infty)$ and for some constant $K > 0$ and $T > 0$ it holds that*

$$|f(t)| \le K e^{at} \text{ for all } t > T, \text{(} f \text{ is said to be of exponential order).}$$

Then the Laplace transform of f exists for $s > a$.

Theorem 4.1.7. *[Behavior at infinity of Laplace transform] Suppose f is exponential order. Then the Laplace transform of f satisfies*

$$\lim_{s\to\infty} \mathcal{L}\{f(t)\} = 0 \tag{4.2}$$

Theorem 4.1.8. *[Linearity of Laplace transform] Suppose the Laplace transform of f_1, f_2 exist for $s > a_1, s > a_2$ resp. Then the Laplace transform of $c_1 f_1 + c_2 f_2$ exists for $s > \max(a_1, a_2)$ and we have*

$$\mathcal{L}\{c_1 f_1(t) + c_2 f_2(t)\} = c_1 \mathcal{L}\{f_1(t)\} + c_2 \mathcal{L}\{f_2(t)\}. \tag{4.3}$$

Proof.

$$\begin{aligned}
\mathcal{L}\{c_1 f_1(t) + c_2 f_2(t)\} &= \int_0^\infty e^{-st}\left[c_1 f_1(t) + c_2 f_2(t)\right] dt \\
&= c_1 \int_0^\infty e^{-st} f_1(t) dt + c_2 \int_0^\infty e^{-st} f_2(t) dt \\
&= c_1 \mathcal{L}\{f_1(t)\} + c_2 \mathcal{L}\{f_2(t)\}.
\end{aligned}$$

□

Example 4.1.9 (Transform of piecewise continuous function)**.** Find $\mathcal{L}\{f(t)\}$, where $f(t) = \begin{cases} 0, & 0 \le t < 3 \\ 2, & t \ge 3. \end{cases}$

Sol.

$$\begin{aligned}
\mathcal{L}\{f(t)\} &= \int_0^\infty e^{-st} f(t) dt \\
&= \int_3^\infty 2e^{-st} dt \\
&= -\frac{2e^{-st}}{s}\Big|_0^\infty \\
&= \frac{2e^{-s}}{s}, \ s > 0.
\end{aligned}$$

Exercise 4.1.10. (1) Find Laplace transform of the following functions.

(a) $f(t) = t$

(b) $f(t) = t^2$

(c) $f(t) = t^5$

(d) $f(t) = \begin{cases} 1 & 0 \le t \le 1 \\ 2 - t & 1 < t \le 2 \\ 2 + t & 2 < t \le 3 \end{cases}$

(e) $f(t) = \begin{cases} t & 0 \le t \le 1 \\ t^2 & 1 < t \le 2 \\ 1 & 2 < t \le 3 \end{cases}$

(f) $f(t) = te^{at}$

(g) $f(t) = t^2 e^{at}$

(h) $f(t) = t^n e^{at}$

(i) $f(t) = t \sin at$

(j) $f(t) = t^2 \sin at$

(k) $f(t) = t^n \sin at$

(l) $f(t) = t^n \cos at$

(2) For $x > 0$ the Gamma function is given by

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\, dt.$$

For $p > -1$ find $\mathcal{L}\{t^p\}$ by Gamma function.

(3) If n is integer show that $\mathcal{L}\{t^n\} = n!/s^{n+1}$ holds.

(4) Show $\mathcal{L}\{t^{-1/2}\} = \sqrt{\pi/s}$.

(5) Find $\mathcal{L}\{t^{1/2}\}$.

$f(t)$	$F(s) = \mathcal{L}\{f(t)\}$	$f(t)$	$F(s) = \mathcal{L}\{f(t)\}$
1	$\frac{1}{s}, \quad s > 0$	e^{at}	$\frac{1}{s-a}, \quad s > a$
$t^n, n > 0$ integer	$\frac{n!}{s^{n+1}}, \quad s > 0$	$t^p, \quad p > -1$	$\frac{\Gamma(p+1)}{s^{p+1}}, \quad s > 0$
$\sin at$	$\frac{a}{s^2+a^2}, \quad s > 0$	$\cos at$	$\frac{s}{s^2+a^2}, \quad s > 0$
$e^{at} \sin bt$	$\frac{b}{(s-a)^2+b^2}, \quad s > a$	$e^{at} \cos bt$	$\frac{s-a}{(s-a)^2+b^2}, \quad s > a$
$t^n e^{at}, n > 0$ integer	$\frac{n!}{(s-a)^{n+1}}, \quad s > a$	$u_c(t)$	$\frac{e^{-cs}}{s}, \quad s > 0$

Table. Laplace transform of important functions

4.2 Inverse transform and transform of derivatives

4.2.1 Inverse transform

Theorem 4.2.1. *[Uniqueness of inverse transform] If we have*

$$\mathcal{L}\{f_1(t)\} = \mathcal{L}\{f_2(t)\},$$

and $f_1(t)$, $f_2(t)$ are continuous, then $f_1(t) = f_2(t)$. Even if $f_1(t)$, $f_2(t)$ are not continuous, $f_1(t) = f_2(t)$ almost everywhere.

Example 4.2.2. $Y(s) = \frac{2}{s-3} + \frac{1}{s^2+4}$ is given. Find $\mathcal{L}^{-1}\{Y\}$.

Sol. We have seen $\mathcal{L}\{e^{3t}\} = \frac{1}{s-3}$, $\mathcal{L}\{\sin 2t\} = \frac{2}{s^2+2^2}$. So

$$\mathcal{L}^{-1}\{Y\} = 2e^{3t} + \frac{1}{2}\sin 2t.$$

by uniqueness.

□

When $Y(s)$ consists of several functions, we may use linearity of inverse transform $Y(s)$.

Theorem 4.2.3. *[Linearity of inverse transform] If $Y(s) = c_1Y_1(s) + \cdots + c_nY_n(s)$, then we have*

$$\mathcal{L}^{-1}\{Y(s)\} = c_1\mathcal{L}^{-1}\{Y_1(s)\} + \cdots + c_n\mathcal{L}^{-1}\{Y_n(s)\}. \tag{4.4}$$

Example 4.2.4. $Y(s) = \frac{-2s+6}{s^2+4}$ is given. Find $\mathcal{L}^{-1}\{Y(s)\}$.

Sol.

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{-2s+6}{s^2+4}\right\} &= \mathcal{L}^{-1}\left\{\frac{-2s}{s^2+4} + \frac{6}{s^2+4}\right\}\\ &= -2\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} + 3\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\}\\ &= -2\cos 2t + 3\sin 2t.\end{aligned}$$

□

In certain cases we can use the partial fraction and linearity to find inverse transform of some functions:

Example 4.2.5. Find $\mathcal{L}^{-1}\left\{\frac{s^2+6s+9}{(s-1)(s-2)(s+4)}\right\}$.

Sol.

$$\frac{s^2+6s+9}{(s-1)(s-2)(s+4)} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s+4}.$$

How to find A, B, C ? Use cover-up method:

$$\left.\frac{s^2+6s+9}{\boxed{\text{(s-1)}}(s-2)(s+4)}\right|_{s=1} = A = -\frac{16}{5}$$

$$\left.\frac{s^2+6s+9}{(s-1)\boxed{\text{(s-2)}}(s+4)}\right|_{s=2} = B = \frac{25}{6}$$

$$\left.\frac{s^2+6s+9}{(s-1)(s-2)\boxed{\text{(s+4)}}}\right|_{s=-4} = C = \frac{1}{30}$$

$$\mathcal{L}^{-1}\left\{\frac{s^2+6s+9}{(s-1)(s-2)(s+4)}\right\} = -\frac{16}{5}e^t + \frac{2}{5}6e^{2t} + \frac{2}{5}6e^{-4t}.$$

$f(t)$	$F(s) = \mathcal{L}\{f(t)\}$	$f(t)$	$F(s) = \mathcal{L}\{f(t)\}$
1	$\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}, \quad s > 0$	t^n	$\mathcal{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\}, \quad s > 0$
$\sin at$	$\mathcal{L}^{-1}\left\{\frac{a}{s^2+a^2}\right\}, \quad s > 0$	$\cos at$	$\mathcal{L}^{-1}\left\{\frac{s}{s^2+a^2}\right\}, \quad s > 0$
$\sinh kt$	$\mathcal{L}^{-1}\left\{\frac{k}{s^2-k^2}\right\}, \quad s > k$	$\cosh kt$	$\mathcal{L}^{-1}\left\{\frac{s}{s^2-k^2}\right\}, \quad s > k$

Table. Laplace inverse transform

4.2.2 Transform of derivatives

Theorem 4.2.6. *If f is continuous on $[0, \infty)$ and f' is piecewise continuous and there exist constants $M > 0$, $T > 0$ and a such that*

$$|f(t)| \le Me^{at}, |f'(t)| \le Me^{at}$$

holds, then the following holds.

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0). \tag{4.5}$$

Proof.

$$\begin{aligned} \int_0^\infty e^{-st} f'(t)dt &= e^{-st} f(t) \mid_0^\infty + s\int_0^\infty e^{-st} f(t)dt \\ &= -f(0) + s\int_0^\infty e^{-st} f(t)dt \\ &= s\mathcal{L}\{f(t)\} - f(0). \end{aligned}$$

□

Repeating above process we obtain the following result.

Theorem 4.2.7. *Suppose $f, f', \cdots, f^{(n-1)}$ are continuous on $[0,\infty)$ and piecewise continuous on $f^{(n)}$ and there exist constants $M > 0$, $T > 0$ and a such that*

$$|f(t)| \le Me^{at}, |f'(t)| \le Me^{at}, \cdots, |f^{(n-1)}(t)| \le Me^{at},\ t > T.$$

Then $\mathcal{L}\{f^{(n)}(t)\}$ exists on $s > a$ and the following holds.

$$\mathcal{L}\{f^{(n)}(t)\} = s^n\mathcal{L}\{f(t)\} - s^{n-1}f(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0) \tag{4.6}$$

According to this Theorem, the Laplace transform transforms a constant coefficient, linear DE. into an algebraic equation in s. Also the coefficients involves all the IC's of IVP. Using this and inverse transform, one can solve IVP. Also note that

$$\text{deg. in } s + \text{order of deriv} = n - 1$$

Assume we have

$$L[y] = a_n y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = g(t). \tag{4.7}$$

Its Laplace transform gives

$$P(s)Y(s) = Q(s) + G(s) \tag{4.8}$$

where $P(s) = a_n s^n + a_{n-1}s^{n-1} + \cdots + a_0$ and $G(s) = \mathcal{L}(g(t))$, and $Q(s)$ comes from the IC's. Then

$$Y(s) = \frac{Q(s)}{P(s)} + \frac{G(s)}{P(s)}.$$

Inverse transform gives

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\{\frac{Q(s)}{P(s)}\} + \mathcal{L}^{-1}\{\frac{G(s)}{P(s)}\} = \mathcal{L}^{-1}\{W(s)Q(s)\} + \mathcal{L}^{-1}\{W(s)G(s)\} \\ &= y_0(t) + y_1(t) \qquad (4.9) \end{aligned}$$

Example 4.2.8. Solve the IVP

$$y'' - 3y' - 4y = 0, \quad y(0) = 1, y'(0) = 2$$

using the Laplace transform.

Sol. Let $\mathcal{L}\{y(t)\} = Y(s)$ and use Laplace transform to get

$$s^2Y(s) - sy(0) - y'(0) - 3[sY(s) - y(0)] - 4Y(s) = 0.$$

Then

$$\begin{aligned} Y(s) &= \frac{s-1}{s^2-3s-4} = \frac{s-1}{(s-4)(s+1)} \\ &= \frac{a}{s-4} + \frac{b}{s+1} = \frac{a(s+1)+b(s-4)}{(s-4)(s+1)}. \end{aligned}$$

Comparing

$$a + b = 1, \quad a - 4b = -1,$$

we get $a = \frac{3}{5}$, $b = \frac{2}{5}$. Hence

$$Y(s) = \frac{3/5}{s-4} + \frac{2/5}{s+1}.$$

Now the remaining task is to find the inverse of $Y(s)$. Since $\mathcal{L}\{e^{ct}\} = \frac{1}{s-c}$ we have by

$$\mathcal{L}^{-1}\left\{\frac{3/5}{s-4} + \frac{2/5}{s+1}\right\} = \frac{3}{5}\mathcal{L}^{-1}\left\{\frac{1}{s-4}\right\} + \frac{2}{5}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}$$

and inverse transform we see $y(t) = \frac{3}{5}e^{4t} + \frac{2}{5}e^{-t}$.

□

Example 4.2.9. Solve IVP $y'' + 4y = e^t, \quad y(0) = 0, y'(0) = 1$.

Solution. Laplace transform gives

$$s^2Y(s) - sy(0) - y'(0) + 4Y(s) = \frac{1}{s-1}, \quad s > 1.$$

Solve it for $Y(s)$ using the IC, we get $(s^2+4)Y(s) = \frac{1}{s-1} + 1$. So

$$Y(s) = \frac{1}{(s-1)(s^2+4)} + \frac{1}{s^2+4}.$$

Use the partial fraction to see the RHS is

$$\frac{A}{s-1} + \frac{Bs+C}{s^2+4} = \frac{(A+B)s^2 + (-B+C)s + 4A - C}{(s-1)(s^2+4)}.$$

Comparing like terms we get $A = \frac{1}{5}, B = -\frac{1}{5}, C = -\frac{1}{5}$. Since

$$Y(s) = \frac{1}{5}\frac{1}{s-1} - \frac{1}{5}\frac{s}{s^2+2^2} + \frac{2}{5}\frac{2}{s^2+2^2},$$

we see

$$y(t) = \frac{1}{5}e^t - \frac{1}{5}\cos 2t + \frac{2}{5}\sin 2t.$$

Exercise 4.2.10. (1) Find the inverse Laplace transform of the following.

(a) $\frac{1}{(s-1)^2}$
(b) $\frac{s}{s^2-3^2}$
(c) $\frac{1}{s^2+3s-2}$
(d) $\frac{s-1}{s^2-2}$
(e) $\frac{1}{s^2+4}$
(f) $\frac{2s-1}{s(s^2+16)}$
(g) $\frac{s+1}{s^2-2s+4}$
(h) $\frac{s+2}{s^2-3s+4}$

(2) Solve IVP.

(a) $y'' - 2y' - 3y = 0, \quad y(0) = 2,\ y'(0) = 2$
(b) $y'' + 5y' + 4y = 0, \quad y(0) = 1,\ y'(0) = 1$
(c) $y'' - 2y' + 3y = 0, \quad y(0) = 1,\ y'(0) = 0$
(d) $y'' + 4y' + 2y = 0, \quad y(0) = 0,\ y'(0) = 1$
(e) $y'' - 3y' + 2y = \sin t, \quad y(0) = 0,\ y'(0) = 1$
(f) $y'' - 2y' + y = e^t, \quad y(0) = 1,\ y'(0) = 3$

(3) Use the Laplace transform to express the following equation in terms of $Y(s)$.

(a) $y'' - 2y = \begin{cases} t, & 0 \le t \le 1 \\ 1, & 1 < t < \infty \end{cases}, \quad y(0) = 1, y'(0) = -1$

(b) $y'' + y = \begin{cases} t, & 0 \le t \le 1 \\ 0, & 1 < t < \infty \end{cases}, \quad y(0) = 1, y'(0) = 0$

(c) $y'' + 3y = \begin{cases} \sin t, & 0 \le \pi \le t \\ \pi, & 1 < t < \infty \end{cases}, \quad y(0) = 1, y'(0) = -1$

(4) Find the Laplace transform of $\sin t$ by Taylor series.

(5) Find the Laplace transform of $\frac{\sin t}{t}$.

4.3 Translation Theorems

4.3.1 Translation on the s-axis

Theorem 4.3.1 (First Translation Theorem)**.** *If $\mathcal{L}\{f(t)\} = F(s)$ and a is a real then*

$$\mathcal{L}\{e^{at} f(t)\} = F(s-a). \tag{4.10}$$

Conversely,

$$e^{at} f(t) = \mathcal{L}^{-1}\{F(s-a)\} \tag{4.11}$$

holds.

Proof. $\mathcal{L}\{e^{at} f(t)\} = \int_0^\infty e^{-st} e^{at} f(t)\,dt = \int_0^\infty e^{-(s-a)t} f(t)\,dt = F(s-a).$ □

Example 4.3.2. Find

$$(a) \cdots \ \mathcal{L}\left\{e^{5t} t^3\right\} \ \textit{and}\ (b) \cdots \ \mathcal{L}\left\{e^{-2t} \cos 4t\right\}.$$

Sol. (a) Since $\mathcal{L}\{t^3\} = \frac{3!}{s^4}$, we have $\mathcal{L}\{e^{5t} t^3\} = \frac{3!}{(s-5)^4}$. (b) Use $\mathcal{L}\{\cos 4t\} = \frac{s}{s^2+16}$ to see

$$\mathcal{L}\left\{e^{-2t} \cos 4t\right\} = \frac{s+2}{(s+2)^2 + 16}.$$

Example 4.3.3. Find

$$\mathcal{L}^{-1}\left\{\frac{2s+5}{(s-3)^2}\right\}.$$

Sol. Note that

$$\frac{2s+5}{(s-3)^2} = \frac{2}{(s-3)} + \frac{11}{(s-3)^2}.$$

Hence

$$\mathcal{L}^{-1}\left\{\frac{2s+5}{(s-3)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{2}{(s-3)}\right\} + \mathcal{L}^{-1}\left\{\frac{11}{(s-3)^2}\right\} = 2e^{3t} + 11e^{3t} t.$$

Example 4.3.4. Find

$$\mathcal{L}^{-1}\left\{\frac{s+1}{s^2-4s+13}\right\}.$$

Sol. Let

$$\begin{aligned} F(s) &:= \frac{s+1}{s^2-4s+13} = \frac{s+1}{(s-2)^2+3^2} = \frac{(s-2)+3}{(s-2)^2+3^2} \\ &= \frac{s-2}{(s-2)^2+3^2} + \frac{3}{(s-2)^2+3^2}. \end{aligned}$$

Last two terms are shifted form of

$$F_1(s) = \frac{s}{s^2+3^2} = \mathcal{L}\{\cos 3t\}, \quad F_2(s) = \frac{3}{s^2+3^2} = \mathcal{L}\{\sin 3t\}.$$

Hence inverse transform of F is

$$e^{2t}\cos 3t + e^{2t}\sin 3t.$$

□

Example 4.3.5 (IVP by Laplace transform)**.** Solve $y'' - 6y' + 9y = t^2e^{3t}$, $y(0) = 2$, $y'(0) = 17$.

Sol. Laplace transform gives

$$s^2Y(s) - sy(0) - y'(0) - 6[sY(s) - y(0)] + 9Y(s) = \frac{2}{(s-3)^3}.$$

Thus

$$\begin{aligned} (s^2-6s+9)Y(s) &= 2s+5+\frac{2}{(s-3)^3} \\ Y(s) &= \frac{2s+5}{(s-3)^2} + \frac{2}{(s-3)^5} \\ &= \frac{2}{s-3} + \frac{11}{(s-3)^2} + \frac{2}{(s-3)^5}. \end{aligned}$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{2}{s-3} + \frac{11}{(s-3)^2} + \frac{2}{(s-3)^5}\right\} = 2e^{3t} + 11te^{3t} + \frac{1}{12}t^4e^{3t}.$$

Example 4.3.6 (IVP by Laplace transform)**.** Solve $y'' + 4y' + 6y = 1 + e^{-t}$, $y(0) = 0$, $y'(0) = 0$.

Sol. Laplace transform gives

$$s^2Y(s) - sy(0) - y'(0) + 4[sY(s) - y(0)] + 6Y(s) = \frac{1}{s} + \frac{1}{s+1}.$$

Thus

$$\begin{aligned}(s^2+4s+6)Y(s) &= \frac{2s+1}{s(s+1)}\\ Y(s) &= \frac{2s+1}{s(s+1)(s^2+4s+6)}.\end{aligned}$$

Partial fraction

$$Y(s) = \frac{1/6}{s} + \frac{1/3}{s+1} + \frac{s/2+5/3}{s^2+4s+6}.$$

$$\frac{s/2+5/3}{s^2+4s+6} = \frac{1}{2}\frac{s+2-2+\frac{10}{3}}{(s+2)^2+2} = \frac{1}{2}\frac{s+2}{(s+2)^2+2} + \frac{2}{3\sqrt{2}}\frac{\sqrt{2}}{(s+2)^2+2}. \quad (4.12)$$

Thus

$$y(t) = \frac{1}{6} + \frac{1}{3}e^{-t} - \frac{1}{2}e^{-2t}\cos\sqrt{2}t - \frac{\sqrt{2}}{3}e^{-2t}\sin\sqrt{2}t.$$

4.3.2 Translation on the t-axis

Unit step function

For a real a,

$$u(t-a) = \begin{cases} 0, & t < a \\ 1, & t \ge a \end{cases}$$

is called the **unit step function**. A piecewise defined function of the form

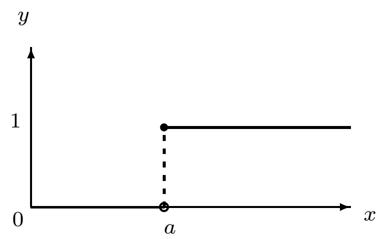


Figure 4.1: The unit step function $u(t-a)$

$$f(t) = \begin{cases} g(t), & 0 \le t < a \\ h(t), & t \ge a. \end{cases} \tag{4.13}$$

is the same as

$$f(t) = g(t) - g(t)u(t-a) + h(t)u(t-a) = g(t) - u(t-a)(g(t) - h(t)). \tag{4.14}$$

Express $f(t) = \begin{cases} 20t, & 0 \le t < 5 \\ 0, & t \ge 5 \end{cases}$ with step functions.

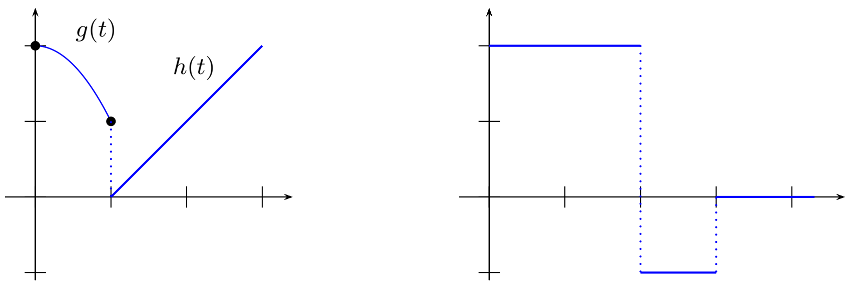


Figure 4.2: $g(t) - g(t)u(t-1) + h(t)u(t-1)$ and $f(t) = 2 - 3u(t-2) + u(t-3)$

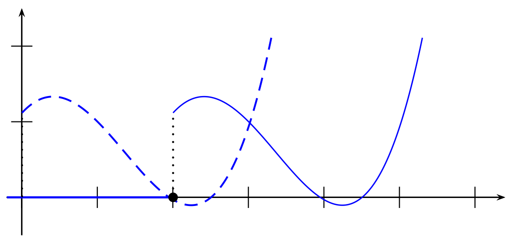


Figure 4.3: $f(x) = xe^{-x} + x^3/6 - x^2$, shifted: $f(t-2)u(t-2)$

Consider a function $f(t)$ shifted by a to the right.

$$\bar{f}(t) = \begin{cases} 0, & t < a \\ f(t-a), & t \ge a. \end{cases}$$

We can simply write it as $u(t-a)f(t-a)$.

Theorem 4.3.7. *[Second Translation Theorem] Assume the Laplace transform of f exist for $s > a_0 \geq 0$ and write it as $F(s) = \mathcal{L}\{f\}$. Then for $a > 0$ we have*

$$\mathcal{L}\{u(t-a)f(t-a)\} = e^{-as}F(s). \tag{4.15}$$

Hence the inverse transform is

$$u(t-a)f(t-a) = \mathcal{L}^{-1}\{e^{-as}F(s)\}. \tag{4.16}$$

Proof. By definition

$$\begin{aligned}
\mathcal{L}\{u(t-a)f(t-a)\} &= \int_0^\infty e^{-st}u(t-a)f(t-a)dt \\
&= \int_a^\infty e^{-st}f(t-a)dt \\
&= \int_0^\infty e^{-s(\tau+a)}f(\tau)d\tau, \quad \tau = t-a \\
&= e^{-as}\int_0^\infty e^{-s\tau}f(\tau)d\tau \\
&= e^{-as}F(s).
\end{aligned}$$

□

Example 4.3.8. Laplace transform of

$$f(t) = \begin{cases} 0, & t < \frac{\pi}{2} \\ \cos(t - \frac{\pi}{2}), & t \geq \frac{\pi}{2} \end{cases}$$

is

$$\mathcal{L}\{f\} = e^{-\frac{\pi s}{2}}\frac{s}{s^2+1},$$

where we used $a = \frac{\pi}{2}$, $\mathcal{L}\{\cos t\} = \frac{s}{s^2+1}$.

Example 4.3.9. Find the Laplace transform of

$$f(t) = \begin{cases} \cos t, & t < \frac{\pi}{2} \\ \cos t + e^{2(t-\frac{\pi}{2})}, & t \geq \frac{\pi}{2}. \end{cases}$$

Sol. Set

$$g(t) = \begin{cases} 0, & t < \frac{\pi}{2} \\ e^{2(t-\frac{\pi}{2})}, & t \geq \frac{\pi}{2}. \end{cases}$$

Then $f(t) = \cos t + g(t)$. Since $g(t) = u(t-\frac{\pi}{2})e^{2(t-\frac{\pi}{2})}$ and $\mathcal{L}\{e^{2t}\} = \frac{1}{s-2}$ we have

$$\mathcal{L}\{f\} = \frac{s}{s^2+1} + \frac{e^{-\frac{\pi s}{2}}}{s-2}.$$

⌟

Inverse form of Theorem 4.3.7

Example 4.3.10. Find (a) $\mathcal{L}^{-1}\{\frac{e^{-2s}}{s-4}\}$ and (b) $\mathcal{L}^{-1}\{\frac{s}{s^2+9}e^{-\pi s/2}\}$.

Sol. (a) Since $\mathcal{L}^{-1}\left\{\frac{1}{s-4}\right\} = e^{4t}$, we have

$$\mathcal{L}^{-1}\{F(s)\} = u(t-2)e^{4(t-2)}.$$

(b) With $a = \pi/2$, $F(s) = \frac{s}{s^2+9}$, $\mathcal{L}^{-1}\{F(s)\} = \cos 3t$. Thus

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}e^{-\pi s/2}\right\} = \cos 3(t-\frac{\pi}{2})u(t-\frac{\pi}{2}).$$

Example 4.3.11. Solve the IVP by Laplace transformation.

$$y'' + y = g(t) = \begin{cases} t, & t \le 2 \\ 2, & t > 2 \end{cases}$$

$$y(0) = y'(0) = 0.$$

Solution. Express the rhs as $g(t) = t - tu(t-2) + 2u(t-2) = t - (t-2)u(t-2)$. Laplace transformation give

$$s^2Y(s) - sy(0) - y'(0) + Y(s) = \frac{1-e^{-2s}}{s^2}.$$

Solving this for $Y(s)$, we get

$$Y(s) = \frac{1-e^{-2s}}{s^2(s^2+1)}.$$

Set $H(s) = \frac{1}{s^2} - \frac{1}{s^2+1}$. Then since

$$\mathcal{L}^{-1}\{H\} = h(t) = t - \sin t,$$

we have shifting theorem

$$y(t) = h(t) - u(t-2)h(t-2) = \begin{cases} t - \sin t, & t \le 2 \\ 2 - \sin t + \sin(t-2), & t > 2. \end{cases}$$

Example 4.3.12. Solve the IVP by Laplace transformation.

$$y'' + 2y' + 2y = u(t-\pi), \quad y(0) = y_0, y'(0) = y_1.$$

Solution. Laplace transformation gives

$$s^2Y(s) - sy(0) - y'(0) + 2[sY(s) - y(0)] + 2Y(s) = \frac{e^{-\pi s}}{s}.$$

$$(s^2 + 2s + 2)Y(s) = \frac{e^{-\pi s}}{s} + sy_0 + 2y_0 + y_1.$$

Hence

$$Y(s) = \frac{e^{-\pi s}}{s(s^2 + 2s + 2)} + \frac{sy_0 + 2y_0 + y_1}{s^2 + 2s + 2}.$$

Using the partial fraction

$$\frac{1}{s(s^2 + 2s + 2)} = \frac{a}{s} + \frac{bs + c}{s^2 + 2s + 2} = \frac{(a + b)s^2 + (2a + c)s + 2a}{s(s^2 + 2s + 2)}$$

we get

$$\begin{aligned} a + b &= 0 \\ 2a + c &= 0 \\ 2a &= 1. \end{aligned}$$

Thus $a = \frac{1}{2}, b = -\frac{1}{2}, c = -1$. By setting

$$F(s) = \frac{1}{2s} + \frac{-\frac{s}{2} - 1}{s^2 + 2s + 2} = \frac{1}{2s} - \frac{1}{2}\frac{s + 1}{(s + 1)^2 + 1} - \frac{1}{2}\frac{1}{(s + 1)^2 + 1},$$

we get

$$f(t) = \mathcal{L}^{-1}\{F\} = \frac{1}{2} - \frac{1}{2}e^{-t}\cos t - \frac{1}{2}e^{-t}\sin t.$$

Also if we set

$$G(s) = \frac{sy_0 + 2y_0 + y_1}{s^2 + 2s + 2} = \frac{(s + 1)y_0 + y_0 + y_1}{(s + 1)^2 + 1},$$

then

$$g(t) = \mathcal{L}^{-1}\{G\} = y_0e^{-t}\cos t + (y_0 + y_1)e^{-t}\sin t.$$

Hence

$$y(t) = u(t - \pi)f(t - \pi) + g(t).$$

Alternative form of Theorem 4.3.7

Recall that Theorem 4.3.7 can be used only if the function is given in the form $f(t-a)u(t-a)$. If we are required to find the Laplace transform of a function like $g(t)u(t-2)$ we have to change it like $(g(t-a) + some\ function)u(t-2)$, which is not easy.

For example we consider the Laplace transform of the function $t^2u(t-2)$. **Method 1**: Use the expansion $t^2 = (t-2)^2 + 4(t-2) + 4$. So

$$\begin{aligned}\mathcal{L}\{t^2u(t-2)\} &= \mathcal{L}\{[(t-2)^2 + 4(t-2) + 4]u(t-2)\} \\ &= e^{-2s}\left(\frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s}\right).\end{aligned}$$

In this case, t^2 was a polynomial so we could write it as $(t-2)^2+4(t-2)+4$ and apply the Theorem. But in general, such a transform is not easy(even impossible!)

Method 2: Alternatively, we can compute the Laplace transform of $g(t)u(t-a)$ as follows.

$$\begin{aligned}\mathcal{L}\{g(t)u(t-a)\} &= \int_0^\infty e^{-st}u(t-a)g(t)dt \\ &= \int_a^\infty e^{-st}g(t)dt \\ &= \int_0^\infty e^{-s(u+a)}g(u+a)du, \quad u = t-a \\ &= e^{-as}\int_0^\infty e^{-su}g(u+a)du \\ &= e^{-as}\mathcal{L}\{g(t+a)\}.\end{aligned}$$

For example if $g(t) = t^2$

$$\begin{aligned}\mathcal{L}\{t^2u(t-2)\} &= e^{-2s}\mathcal{L}\{(t+2)^2\} \\ &= e^{-2s}\mathcal{L}\{t^2 + 4t + 4\} \\ &= e^{-2s}\left(\frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s}\right)\end{aligned}$$

Example 4.3.13. Solve $y' + y = f(t) = \begin{cases} 0, & 0 \le t < \pi \\ 3\cos t, & t \ge \pi. \end{cases}$, $y(0) = 5$,

We have $f(t) = 3\cos t \cdot u(t-\pi)$. Using the method above, we see the Laplace transform of $f(t)$ becomes

$$3\mathcal{L}\{\cos t \cdot u(t-\pi)\} = 3e^{-\pi s}\mathcal{L}\{\cos(t+\pi)\} = -3e^{-\pi s}\mathcal{L}\{\cos t\} = -3e^{-\pi s}\frac{s}{s^2+1}.$$

Hence the DE. become

$$sY(s) - y(0) + Y(s) = -e^{-\pi s}\frac{3s}{(s^2+1)} \tag{4.17}$$

$$(s+1)Y(s) = 5 - e^{-\pi s}\frac{3s}{(s^2+1)} \tag{4.18}$$

$$Y(s) = \frac{5}{s+1} - \frac{3e^{-\pi s}}{2}\left[\frac{1}{s+1} + \frac{1}{s^2+1} + \frac{s}{s^2+1}\right]. \tag{4.19}$$

The inverse transform gives

$$\mathcal{L}^{-1}\left\{e^{-\pi s}\frac{1}{s+1}\right\} = e^{-(t-\pi)}u(t-\pi), \quad \mathcal{L}^{-1}\left\{e^{-\pi s}\frac{1}{s^2+1}\right\} = \sin(t-\pi)u(t-\pi)$$

and

$$\mathcal{L}^{-1}\left\{e^{-\pi s}\frac{s}{s^2+1}\right\} = \cos(t-\pi)u(t-\pi).$$

Hence

$$\begin{aligned} y(t) &= 5e^{-t} + \frac{3}{2}e^{-(t-\pi)}u(t-\pi) - \frac{3}{2}\sin(t-\pi)u(t-\pi) - \frac{3}{2}\cos(t-\pi)u(t-\pi) \\ &= 5e^{-t} + \frac{3}{2}\left[e^{-(t-\pi)} - \sin(t-\pi) - \cos(t-\pi)\right]u(t-\pi) \\ &= 5e^{-t} + \frac{3}{2}\left[e^{-(t-\pi)} + \sin t + \cos t\right]u(t-\pi). \end{aligned}$$

Skip Example 10.

Exercise 4.3.14. (1) Find the inverse transform of

(a) $\frac{e^{-2s}+e^{-s}}{s^2}$

(b) $\frac{e^{-2s}}{s^2+2s+5}$

(c) $\frac{e^{-3s}}{s-1}$

(d) $\frac{e^{-s}}{s^2+4}$

(e) $\frac{se^{-2s}}{s^2+1}$

(f) $\frac{1-e^{-2s}}{s^2+1}$

(2) Find the Laplace transform of

(a) $f(t) = u_1(t) - u_2(t) + 3u_5(t)$

(b) $f(t) = \begin{cases} 0, & 0 \le t \le 1 \\ (t-1)^2, & 1 < t < \infty \end{cases}$

(c) $f(t) = (t-2)u_3(t) + (t-1)u_2(t)$

(3) Find the Laplace transform of a periodic function $f(t+T) = f(t)$, $0 \le t < \infty$. Find the Laplace transform of the following.

(a) $f(t) = \begin{cases} 1, & 0 \le t \le \pi \\ -1, & \pi < t \le 2\pi \end{cases}, \quad f(t+2\pi) = f(t)$

(b) $f(t) = \begin{cases} 1, & 0 \le t \le 1 \\ 0, & 1 < t \le 2 \end{cases}, \quad f(t+2) = f(t)$

(c) $f(t) = \cos t, 0 \le t \le \pi, \quad f(t+\pi) = f(t)$

(4) Solve the IVP.

(a) $y'' + 2y' + y = f(t), \quad y(0) = 1, y'(0) = 0,$

$$f(t) = \begin{cases} 1, & 0 \le t \le 1 \\ 0, & 1 < t < \infty. \end{cases}$$

(b) $y'' + 4y = f(t), \quad y(0) = 1, y'(0) = 0,$

$$f(t) = u_\pi(t) - u_{2\pi}(t).$$

(c) $y'' + 2Ay' + By = 0, \quad y(0) = a, y'(0) = b.$

4.4 Additional Operational Properties

4.4.1 Derivatives of Laplace transform

Theorem 4.4.1. *If $F(s) = \mathcal{L}\{f(t)\}$, then*

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s).$$

Proof. For $n = 1$, the Laplace transform of f is

$$F(s) = \int_0^\infty e^{-st} f(t) dt. \tag{4.20}$$

Derivative of

$$F'(s) = -\int_0^\infty e^{-st}[tf(t)]dt$$

gives

$$\mathcal{L}\{tf(t)\} = -F'(s).$$

Repeating we obtain the results. □

Example 4.4.2. Noting that the derivative of $\frac{1}{(s^2+a^2)}$ is $-\frac{2s}{(s^2+a^2)^2}$, find the inverse transform of

$$\frac{s}{(s^2+a^2)^2}.$$

We have by Theorem 4.4.1

$$\frac{s}{(s^2+a^2)^2} = -\frac{1}{2}\frac{d}{ds}\frac{1}{(s^2+a^2)} = \frac{1}{2a}\mathcal{L}\{t\sin at\}.$$

Hence

$$\mathcal{L}^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = -\frac{1}{2}\mathcal{L}^{-1}\left\{\frac{d}{ds}\frac{1}{(s^2+a^2)}\right\} = \frac{1}{2a}t\sin at.$$

Example 4.4.3 (IVP)**.** Solve

$$x'' + 16x = \cos 4t, \; x(0) = 0, \; x'(0) = 1.$$

Sol. Transformation and I.C.s give

$$(s^2+16)X(s) = 1 + \frac{s}{s^2+16}, \; \textit{thus } X(s) = \frac{1}{s^2+16} + \frac{s}{(s^2+16)^2}.$$

We have just seen above that

$$\mathcal{L}^{-1}\left\{\frac{s}{(s^2+16)^2}\right\} = \frac{1}{8}t\sin 4t.$$

$$x(t) = \frac{1}{4}\sin 4t + \frac{1}{8}t\sin 4t.$$

Example 4.4.4. Find the inverse transform of

$$F(s) = \ln\frac{s}{s-1}.$$

Hint: $F'(s)$ is easy to handle.

Sol. We use Theorem 4.4.1. Since

$$\mathcal{L}\{tf(t)\} = -F'(s) = -\frac{1}{s} + \frac{1}{s-1}.$$

$$tf(t) = -\mathcal{L}^{-1}\{F'(s)\} = e^t - 1.$$

We have

$$f(t) = \frac{e^t-1}{t}.$$

□

4.4.2 Laplace transform of integrals

Convolution

Given two functions f, g, if the new function $h(t)$ defined by

$$h(t) = (f * g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau$$

exists, then $h(t)$ is called te **convolution** of f and g.

Remark 4.4.5. Meaning of convolution is taking the weighted average of a function f using g. If $g = 1$, $h(t)$ is the average of f between 0 and t.

Theorem 4.4.6. *[Properties of convolution]*

(1) $f * g = g * f$

(2) $f * (g_1 + g_2) = f * g_1 + f * g_2$

(3) $(f * g) * h = f * (g * h)$

(4) $f * 0 = 0 * f = 0.$

Caution: $g * 1 = g$ does not holds. eg., $g(t) = e^t$.

$$\int_0^t e^{t-\tau} \cdot 1 d\tau = -e^{t-\tau}\big|_0^t = -(1 - e^t).$$

$$\int_0^t 1 \cdot e^{\tau} d\tau = e^t.$$

Thus $g * 1 \neq g$.

Theorem 4.4.7. *[convolution Theorem] Let $F(s) = \mathcal{L}\{f\}$, $G(s) = \mathcal{L}\{g\}$ be Laplace transform for f, g. Then*

$$\mathcal{L}\{f * g\} = \mathcal{L}\{f\} \cdot \mathcal{L}\{g\}. \tag{4.21}$$

In other words, if $H(s)$ is the Laplace transform of $h := f * g$, we have

$$H(s) = F(s) \cdot G(s).$$

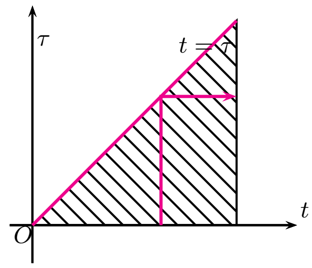


Figure 4.4: Change order of integration

Proof. Since $F(s) = \int_0^\infty e^{-s\tau} f(\tau)d\tau, \quad G(s) = \int_0^\infty e^{-s\eta} g(\eta)d\eta$ we have

$$F(s)G(s) = \int_0^\infty f(\tau)d\tau \int_0^\infty e^{-s(\tau+\eta)} g(\eta)d\eta.$$

Substitute $\tau + \eta = t$. Then

$$F(s)G(s) = \int_0^\infty f(\tau)d\tau \int_\tau^\infty e^{-st} g(t-\tau)dt.$$

Change the order of integration $F(s)G(s)$,

$$\begin{aligned} &= \int_0^\infty e^{-st}dt \int_0^t f(\tau)g(t-\tau)d\tau \\ &= \int_0^\infty e^{-st} h(t)dt = \mathcal{L}\{h\} = H(s). \end{aligned}$$

□

Inverse of convolution Theorem

Write (4.21) in another way:

Theorem 4.4.8. *[Inverse transform of convolution]*

$$\mathcal{L}^{-1}\{F(s)G(s)\} = (f * g)(t).$$

Example 4.4.9. Find

$$\mathcal{L}\left\{\int_0^t e^\tau \sin(t-\tau)d\tau\right\}.$$

We see

$$\mathcal{L}\left\{\int_0^t e^\tau \sin(t-\tau)d\tau\right\} = \mathcal{L}\{e^t\}\mathcal{L}\{\sin t\} = \frac{1}{s-1} \cdot \frac{1}{s^2+1}.$$

Example 4.4.10. Given

$$H(s) = \frac{a}{s^3(s^2+a^2)}$$

find $\mathcal{L}^{-1}\{H(s)\}$.

Sol. Set

$$F(s) = \frac{1}{s^3}, \quad G(s) = \frac{a}{s^2+a^2}.$$

Since $f(t) = \frac{t^2}{2}$, $g(t) = \sin at$ we have by convolution theorem

$$h(t) = (f * g)(t) = \int_0^t \frac{(t-\tau)^2}{2} \sin a\tau \, d\tau.$$

□

Example 4.4.11. Find inverse transform of

$$F(s) = \frac{1}{(s^2+a^2)^2}.$$

Use inverse of convolution Theorem. Let $F(s) = G(s) = \dfrac{1}{s^2+a^2}$ so that

$$f(t) = g(t) = \frac{1}{a} \sin at.$$

Thus

$$\mathcal{L}^{-1}\{F(s)G(s)\} = f * g = \frac{1}{a^2} \int_0^t \sin a\tau \sin a(t-\tau) dt.$$

Use trig. identity:

$$\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)].$$

$$\begin{aligned}
\frac{1}{a^2} \int_0^t \sin a\tau \sin a(t-\tau) dt &= \frac{1}{2a^2} \int_0^t [\cos a(2\tau - t) - \cos at] d\tau \\
&= \frac{1}{2a^2} \left[\frac{1}{2a} \sin a(2\tau - t) - \tau \cos at \right]_0^t \\
&= \frac{\sin at - a\tau \cos at}{2a^3}.
\end{aligned}$$

Example 4.4.12. Solve the IVP

$$\begin{aligned} y'' - 9y &= r(t), \\ y(0) &= 1, \\ y'(0) &= 2. \end{aligned}$$

Sol. Laplace transform gives

$$s^2 Y(s) - s - 2 - 9Y(s) = R(s).$$

Solve it for $Y(s)$ and use partial fraction to get

$$Y(s) = \frac{s+2}{s^2-9} + \frac{R(s)}{s^2-9} = \frac{5}{6(s-3)} + \frac{1}{6(s+3)} + \frac{R(s)}{s^2-9}.$$

Hence

$$y(t) = \frac{5}{6}e^{3t} + \frac{1}{6}e^{-3t} + \frac{1}{3}\int_0^t \sinh 3(t-\tau) r(\tau) d\tau.$$

Transform of an integral

Theorem 4.4.13. *Suppose f is piecewise-continuous and $|f(t)| \le Me^{at}$ holds, then*

$$\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{1}{s}\mathcal{L}\{f(t)\}, \quad s > 0. \tag{4.22}$$

Proof. Let

$$g(t) = \int_0^t f(\tau)d\tau.$$

Then

$$|g(t)| \le \int_0^t |f(\tau)|d\tau \le M\int_0^t e^{a\tau}d\tau = \frac{M}{a}(e^{a\tau} - 1)$$

and whenever f is continuous $g'(t) = f(t)$ holds and so g' is piecewise-continuous. By Theorem 4.1.6 the Laplace transform of g and g' exist and it holds that

$$\mathcal{L}\{f\} = \mathcal{L}\{g'\} = s\mathcal{L}\{g\} - g(0) = s\mathcal{L}\{g\}.$$

So we have

$$\mathcal{L}^{-1}\left\{\frac{1}{s}F(s)\right\} = g(t) = \int_0^t f(\tau)d\tau. \tag{4.23}$$

□

Remark: This Theorem is also a special case of convolution theorem with $g(t) = 1$.

Example 4.4.14. Find $\mathcal{L}^{-1}\{F\}$ when

$$F(s) = \frac{1}{s(s^2 + a^2)}.$$

Sol. Since

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2 + a^2)}\right\} = \frac{1}{a}\sin at,$$

we have

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s^2 + a^2)}\right\} = \frac{1}{a}\int_0^t \sin a\tau d\tau = \frac{1 - \cos at}{a^2}.$$

□

Example 4.4.15. Find $\mathcal{L}^{-1}\{F\}$ when

$$F(s) = \frac{1}{s^2(s^2 + a^2)}.$$

Sol. Let $G(s) = \frac{1}{s(s^2+a^2)}$. Then

$$F(s) = \frac{1}{s^2(s^2 + a^2)} = \frac{1}{s}G(s)$$

and from Example 4.4.14, we know that $\mathcal{L}^{-1}\{G\} = \frac{1-\cos at}{a^2}$, and so

$$\begin{aligned}\mathcal{L}^{-1}\{F(s)\} &= \mathcal{L}^{-1}\left\{\frac{1}{s}G(s)\right\} = \int_0^t g(\tau)d\tau \\ &= \frac{1}{a^2}\int_0^t (1 - \cos a\tau)d\tau = \frac{1}{a^2}\left(t - \frac{\sin at}{a}\right).\end{aligned}$$

□

Example 4.4.16. Find

$$\mathcal{L}^{-1}\left\{\frac{1}{s^3(s^2 + 1)}\right\}.$$

Sol.

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{1}{s^3(s^2+1)}\right\} &= \mathcal{L}^{-1}\left\{\frac{1}{s}\cdot\frac{1}{s^2(s^2+1)}\right\}\\ &= \int_0^t (\tau-\sin\tau)d\tau\\ &= \frac{t^2}{2}-1+\cos t.\end{aligned}$$

Volterra integral equation

An equation with unknown function under the integral sign is called an integral equation :

$$f(t) = g(t) + \int_0^t f(\tau)h(t-\tau)d\tau.$$

Example 4.4.17 (An integral equation)**.** Solve for $f(t)$:

$$f(t) = 3t^2 - e^{-t} - \int_0^t f(\tau)e^{t-\tau}d\tau.$$

Sol. By convolution Theorem, the Laplace transform of $\int_0^t f(\tau)e^{t-\tau}d\tau$ is the product of the Laplace transform of $f(t)$ and e^t. So

$$F(s) = \frac{6}{s^3} - \frac{1}{s+1} - F(s)\frac{1}{s-1}.$$

$$F(s) = \frac{6}{s^3} - \frac{6}{s^4} + \frac{1}{s} - \frac{2}{s+1}.$$

So the inverse transform gives

$$f(t) = 3t^2 - t^3 + 1 - 2e^{-t}.$$

□

Series circuit

Kirchhoff's law states that the sum of the voltage drops across a resistor, inductor(coil), capacitor(condenser) is equal to the input voltage. We have

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int_0^t i(\tau)d\tau = E(t). \tag{4.24}$$

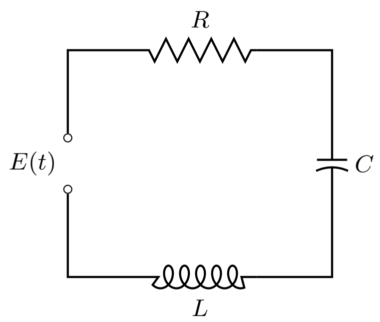


Figure 4.5: RLC-circuit

Example 4.4.18. [RLC**-circuit**] Find the current of the circuit when $L = 0.1h$, $R = 2\Omega$, $C = 0.1f, i(0) = 0$ and $E(t) = 120t - 120tu(t-1)$.

Sol. Using the data we see the integral equation (4.24) becomes

$$0.1\frac{di}{dt} + 2i + 10\int_0^t i(\tau)d\tau = 120t - 120tu(t-1).$$

Let $I(s)$ be the Laplace transform of $i(t)$. Then using the identity $tu(t-1) = (t-1)u(t-1) + u(t-1)$ and translation theorem,

$$0.1sI(s) + 2I(s) + 10\frac{I(s)}{s} = 120\left[\frac{1}{s^2} - \frac{1}{s^2}e^{-s} - \frac{1}{s}e^{-s}\right].$$

Multiplying $10s$,

$$I(s) = 1200\left[\frac{1}{s(s+10)^2} - \frac{1}{s(s+10)^2}e^{-s} - \frac{1}{(s+10)^2}e^{-s}\right].$$

After partial fraction,

$$\begin{aligned} I(s) = 1200 \Bigg[& \frac{1/100}{s} - \frac{1/100}{s+10} - \frac{1/10}{(s+10)^2} - \frac{1/100}{s}e^{-s} \\ & + \frac{1/100}{s+10}e^{-s} + \frac{1/10}{(s+10)^2}e^{-s} - \frac{1}{(s+10)^2}e^{-s}\Bigg]. \end{aligned}$$

Thus

$$\begin{aligned} i(t) = & 12[1 - u(t-1)] - 12[e^{-10t} - e^{-10(t-1)}u(t-1)] \\ & - 12te^{-10t} - 1080(t-1)e^{-10(t-1)}u(t-1). \end{aligned}$$

Return(Redux) of Green's Function

Applying the Laplace transform to

$$y'' + ay' + by = f(t),\ y(0) = 0,\ y'(0) = 0,$$

where a, b are constant, we get

$$Y(s) = \frac{F(s)}{s^2 + as + b}.$$

Here $F(s) = \mathcal{L}\{f(t)\}$. So by inverse form of convolution theorem,

$$y(t) = \int_0^t g(t-\tau)f(\tau)d\tau. \tag{4.25}$$

Here $g(t)$ is the inverse transform of $\frac{1}{s^2+as+b}$:

$$g(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2 + as + b}\right\}.$$

On the other hand from (3.71) of Section 3.10 the solution of IVP is also given by

$$y(t) = \int_0^t G(t,\tau)f(\tau)d\tau, \tag{4.26}$$

where $G(t, \tau)$ is the Green's function. Comparing (4.25) and (4.26) we conclude that

$$G(t, \tau) = g(t - \tau),$$

i.e., we have found the Green's function for this problem. For example, for $y'' + 4y = f(t),\ y(0) = 0,\ y'(0) = 0$ we find

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2 + 4}\right\} = \frac{1}{2}\sin 2t = g(t).$$

So the Green's function is

$$G(t, \tau) = g(t - \tau) = \frac{1}{2}\sin 2(t - \tau).$$

4.4.3 Transform of a periodic function

Theorem 4.4.19 (Transform of a periodic function)**.** *If $f(t)$ is a piecewise continuous function which are periodic with period T, then*

$$\mathcal{L}\{f(t)\} = \frac{1}{1 - e^{-sT}}\int_0^T e^{-st}f(t)dt.$$

Proof. Write

$$\mathcal{L}\{f(t)\} = \int_0^T e^{-st} f(t)dt + \int_T^\infty e^{-st} f(t)dt. \tag{4.27}$$

With $t = u + T$ we have

$$\int_T^\infty e^{-st} f(t)dt = \int_0^\infty e^{-s(u+T)} f(u+T)du = e^{-sT} \int_0^\infty e^{-su} f(u)du = e^{-sT}\mathcal{L}\{f(t)\}. \tag{4.28}$$

Substitute into (4.27) we get

$$\mathcal{L}\{f(t)\} = \int_0^T e^{-st} f(t)dt + \int_T^\infty e^{-st} f(t)dt \tag{4.29}$$

$$= \int_0^T e^{-st} f(t)dt + e^{-sT}\mathcal{L}\{f(t)\}. \tag{4.30}$$

Thus we get the result. □

Example 4.4.20. Find the Laplace transform a periodic function given in the Figure 4.6. On $[0, 2]$, $E(t)$ is defined by

$$E(t) = \begin{cases} 1, & 0 \le t < 1 \\ 0, & 1 \le t < 2 \end{cases}$$

and $E(t + 2) = E(t)$ outside $[0, 2]$.

$$\begin{aligned}\mathcal{L}\{E(t)\} &= \frac{1}{1-e^{-2s}} \int_0^2 e^{-st} E(t)dt = \frac{1}{1-e^{-2s}} \left[\int_0^1 e^{-st} dt\right] \\ &= \frac{1}{1-e^{-2s}} \cdot \frac{1-e^{-s}}{s} \\ &= \frac{1}{s(1+e^{-s})}.\end{aligned}$$

Example 4.4.21. [LR**-circuit with periodic voltage** $E(t)$] Find the current of the LR-circuit when $E(t)$ given periodically as in Figure 4.6. Find the current i when $i(0) = 0$.

Sol. The DE. is

$$L\frac{di}{dt} + Ri = E(t).$$

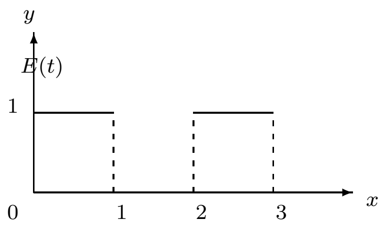


Figure 4.6: Periodic function of Example 4.4.20

Let $F(s)$ be the Laplace transform of $i(t)$. Then

$$LsF(s) + RF(s) = \frac{1}{s(1+e^{-s})} \quad \text{or} \quad F(s) = \frac{1}{L}\frac{1}{s(s+R/L)} \cdot \frac{1}{s(1+e^{-s})}. \tag{4.31}$$

Use the geometric series

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots + \quad \text{or} \quad \frac{1}{1+e^{-s}} = 1 - e^{-s} + e^{-2s} - e^{-3s} + \cdots$$

Since

$$\frac{1}{L}\frac{1}{s(s+R/L)} = \frac{1}{R}\left(\frac{1}{s} - \frac{1}{s+R/L}\right)$$

$$\begin{aligned} F(s) &= \frac{1}{R}\left(\frac{1}{s} - \frac{1}{s+R/L}\right)(1 - e^{-s} + e^{-2s} - e^{-3s} + \cdots) \\ &= \frac{1}{R}\left(\frac{1}{s} - \frac{e^{-s}}{s} + \frac{e^{-2s}}{s} - \frac{e^{-3s}}{s} + \cdots\right) \\ &\quad - \frac{1}{R}\left(\frac{1}{s+R/L} - \frac{e^{-s}}{s+R/L} + \frac{e^{-2s}}{s+R/L} - \frac{e^{-3s}}{s+R/L} + \cdots\right). \end{aligned}$$

Use the second translation theorem

$$\begin{aligned} i(t) &= \frac{1}{R}(1 - u(t-1) + u(t-2) - u(t-3) + \cdots) \\ &\quad - \frac{1}{R}\left(e^{-Rt/L} - e^{-R(t-1)/L}u(t-1) + e^{-R(t-2)/L}u(t-2) - e^{-R(t-3)/L}u(t-3) + \cdots\right) \end{aligned}$$

Thus ...

Exercise 4.4.22. (1) Find the Laplace transform of the following:

(a) $t^2 \cos t$
(b) $te^{-t} \sin t$
(c) $te^{-at} \sin t$
(d) $tu_c(t)$
(e) $t^n \cos bt$
(f) $t^n \sin bt$

(2) Find the Laplace inverse transform of the following:

(a) $\frac{1}{(s+1)^2 s^2}$
(b) $\frac{2s}{(s^2+4)(s^2+9)}$
(c) $\frac{1}{(s^2+a^2)^2}$
(d) $\frac{1}{s^2(s^2+4)(s^2+9)}$

(3) Solve the IVP

(a) $y'' + 4y = r(t), \quad y(0) = 1, y'(0) = 0$
(b) $y'' + 4y = r(t), \quad y(0) = y_0, y'(0) = y_1$
(c) $y'' + 3y' + 2y = \sin t, \quad y(0) = 0, y'(0) = 1$
(d) $y'' + 2y' + 3y = u_c(t), \quad y(0) = 0, y'(0) = 1$

(4) **Volterra integral equation**.

$$f(t) = g(t) + \int_0^t f(\tau)k(t - \tau)\, d\tau.$$

Use Laplace transform to solve the DE.

(a) $f(t) = t - e^t - \int_0^t f(\tau) \sin(t - \tau)\, d\tau$
(b) $f(t) = 2t - \int_0^t (t - \tau) f(\tau)\, d\tau$
(c) $f(t) = te^t + \int_0^t \tau f(t - \tau)\, d\tau$

(5) Solve the following **integro-differential equation**.

(a) $10\frac{dI}{dt} + I + \int_0^t I(\tau)\, d\tau = 100 \sin \pi t, \quad I(0) = 0$
(b) $f'(t) = 1 + t - \int_0^t f(\tau)\, d\tau, \quad f(0) = 1$
(c) $f'(t) = 2t - \int_0^t (t - \tau) f(\tau)\, d\tau, \quad f(0) = 0$

4.5 Dirac delta Function

Impulse function: closing an electric circuit, lightning, crash of cars, ...

Consider Consider the force described by the following function:

$$d_a(t) = \begin{cases} \frac{1}{2a}, & -a \le t \le a \\ 0, & \text{otherwise.} \end{cases}$$

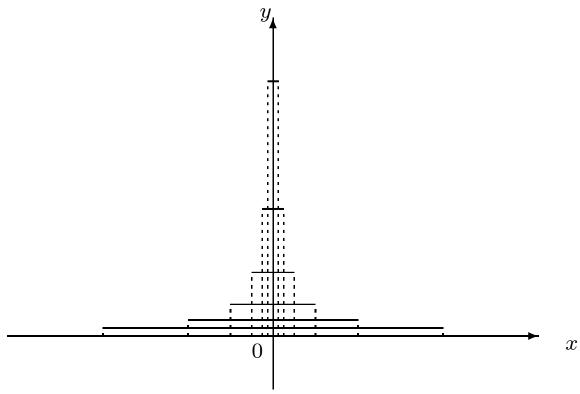


Figure 4.7: function $d_a(t)$ as $a \to 0$

The total energy of the impact is given by

$$I = \int_{-\infty}^{\infty} d_a(t)dt = \int_{-a}^{a} \frac{1}{2a} dt = 1.$$

Let $a \to 0$. The limit does not exists in the usual sense. However, the values 0 for $t \neq t_0$ and when $t = t_0$ the function value cannot exist. So if we define

$$\lim_{a \to 0} d_a(t - t_0) := \delta(t - t_0).$$

We observe the following:

$$\begin{aligned} \delta(t - t_0) &= 0, \quad t \neq t_0 \\ \int_{-\infty}^{\infty} \delta(t - t_0)dt &= 1. \end{aligned}$$

We denote it by $\delta(t - t_0)$ and call it **impulse function** or **Dirac delta function**.
Interpretation: when $f(t)$ is continuous we have

$$\int_{-\infty}^{\infty} \delta(t - t_0)f(t)dt = \lim_{a \to 0} \int_{-\infty}^{\infty} d_a(t - t_0)f(t)dt = \lim_{a \to 0} \frac{1}{2a} \int_{t_0 - a}^{t_0 + a} f(t)dt = f(t_0).$$

Laplace transform of Dirac delta function

Let $t_0 > 0$. Express $d_a(t)$ using the step function, we see

$$d_a(t) = \frac{1}{2a}[u(t + a) - u(t - a)].$$

Consider its shift

$$d_a(t-t_0) = \frac{1}{2a}[u(t-(t_0-a)) - u(t-(t_0+a))].$$

Now the Laplace transform of Dirac delta function is the limit of Laplace transform $d_a(t-t_0)$, i.e.,

$$\begin{aligned}
\mathcal{L}\{\delta(t-t_0)\} &= \lim_{a\to 0}\mathcal{L}\{d_a(t-t_0)\} \\
&= \lim_{a\to 0}\frac{1}{2a}\int_{t_0-a}^{\infty} e^{-st}u(t-(t_0-a))dt - \int_{t_0+a}^{\infty} e^{-st}u(t-(t_0+a))dt \\
&= \lim_{a\to 0}\frac{1}{2a}\int_{t_0-a}^{\infty} e^{-st}dt - \int_{t_0+a}^{\infty} e^{-st}dt \\
&= \lim_{a\to 0}\frac{1}{2sa}\left(e^{-s(t_0-a)} - e^{-s(t_0+a)}\right) \\
&= e^{-t_0 s}\lim_{a\to 0}\frac{1}{2as}\left(e^{sa} - e^{-sa}\right) \\
&= e^{-t_0 s}.
\end{aligned}$$

Theorem 4.5.1. *We have for $t_0 > 0$,*

$$\boxed{\mathcal{L}\{\delta(t-t_0)\} = e^{-t_0 s}}. \tag{4.32}$$

When $t_0 = 0$, it is plausible to set

$$\mathcal{L}\{\delta(t)\} = 1.$$

Thus the behavior is different from the usual case that $\lim_{s\to\infty}\mathcal{L}\{f(t)\} = 0$.

Example 4.5.2. Solve the IVP. (Spring with relatively smaller damping)

$$y'' + 2y' + 2y = 5\delta(t-1), \quad y(0) = y'(0) = 0.$$

Sol. Laplace transform gives

$$s^2Y(s) - sy(0) - y'(0) + 2[sY(s) - y(0)] + 2Y(s) = 5e^{-s}.$$

From this we have

$$Y(s) = 5\frac{e^{-s}}{(s+1)^2+1}$$

and

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+1}\right\} = e^{-t}\sin t.$$

Thus we get

$$y(t) = 5u(t-1)e^{-(t-1)}\sin(t-1).$$

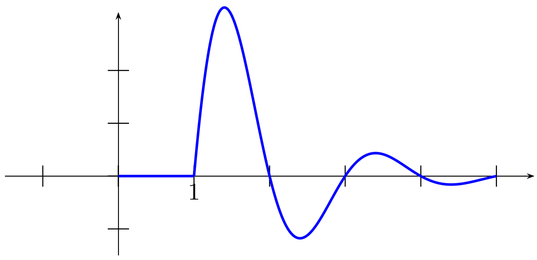


Figure 4.8: Spring movement after impact, $5u(t-1)e^{-(t-1)}\sin(t-1)$

Example 4.5.3. Spring with larger damping.

$$y'' + 3y' + 2y = 5\delta(t-1), \quad y(0) = y'(0) = 0.$$

You may imagine a spring in the heavy viscous oil.

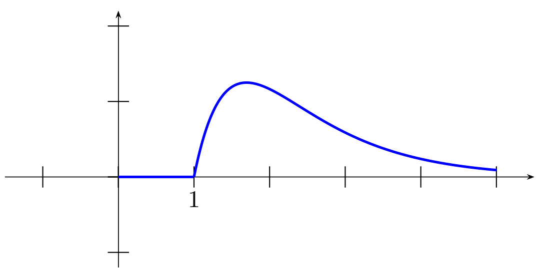


Figure 4.9: Large friction overrides oscillation

Sol. Laplace transform

$$s^2Y(s) - sy(0) - y'(0) + 3[sY(s) - y(0)] + 2Y(s) = 5e^{-s}.$$

Hence

$$Y(s) = \frac{5e^{-s}}{s^2 + 3s + 2}.$$

Inverse transform

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+1)(s+2)}\right\} = e^{-t} - e^{-2t}$$

gives

$$y(t) = 5u(t-1)(e^{-(t-1)} - e^{-2(t-1)}).$$

See figure 4.9.

□

Exercise 4.5.4. (1) Solve IVP

(a) $y'' + 3y' + 2y = \delta(t-1), \quad y(0) = 1, y'(0) = -1$

(b) $y'' + 4y' + 2y = \delta(t-1) + \delta(t-2), \quad y(0) = 0, y'(0) = -1$

(c) $y'' - y = \delta(t - \pi), \quad y(0) = 1, y'(0) = 0$

(d) $y'' + 2y' + 3y = \delta(t-\pi) - \delta(t-2\pi), \quad y(0) = 0, y'(0) = 1$

(e) $y'' + 2y' + 4y = \delta(t-2), \quad y(0) = 1, y'(0) = 0$

4.6 System of linear differential equations

Example 4.6.1. Solve the system of linear differential eq:

$$x_1'' + 10x_1 - 4x_2 = 0$$
$$-4x_1 + x_2'' + 4x_2 = 0$$

subject to $x_1(0) = 0,\ x_1'(0) = 1,\ x_2(0) = 0,\ x_2'(0) = -1$.

Sol. Use Laplace transform.

$$\begin{aligned} s^2X_1(s) - sx_1(0) - x_1'(0) + 10X_1(s) - 4X_2(s) &= 0 \\ -4X_1(s) + s^2X_2(s) - sx_2(0) - x_2'(0) + 4X_2(s) &= 0. \end{aligned}$$

This reduces to

$$\begin{aligned} (s^2 + 10)X_1(s) - 4X_2(s) &= 1 \\ -4X_1(s) + (s^2 + 4)X_2(s) &= -1. \end{aligned}$$

This is an algebraic system of equations in unknowns $X_1(s)$ and $X_2(s)$. Solving this for $X_1(s)$ we get

$$X_1(s) = \frac{s^2}{(s^2+2)(s^2+12)} = -\frac{1/5}{s^2+2} + \frac{6/5}{s^2+12}.$$

Therefore

$$X_1(s) = -\frac{1}{5\sqrt{2}}\frac{\sqrt{2}}{s^2+2} + \frac{6}{5\sqrt{12}}\frac{\sqrt{12}}{s^2+12}.$$

Hence

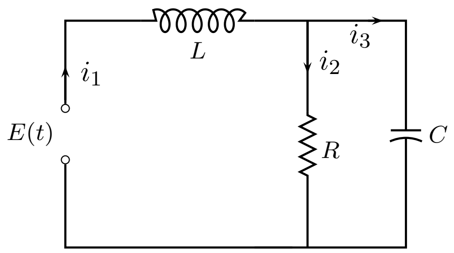
$$x_1(t) = -\frac{\sqrt{2}}{10}\sin\sqrt{2}t + \frac{\sqrt{3}}{5}\sin 2\sqrt{3}t.$$

Substitute $X_1(s)$ to solve for $X_2(s)$ we get

$$X_2(s) = -\frac{s^2+6}{(s^2+2)(s^2+12)} = -\frac{2/5}{s^2+2} + \frac{3/5}{s^2+12}.$$

Proceeding, we get

$$x_2(t) = -\frac{\sqrt{2}}{5}\sin\sqrt{2}t - \frac{\sqrt{3}}{10}\sin 2\sqrt{3}t.$$



Double elec.-circuit

Example 4.6.2 (Elec. Network)**.** Solve the system of linear differential eq:

$$\begin{aligned}\frac{di_1}{dt} + 50i_2 = 60\\ \frac{50}{10^4}\frac{di_2}{dt} + i_2 - i_1 = 0\end{aligned}$$

subject to $i_1(0) = 0, \quad i_2(0) = 0.$

Sol. Use Laplace transform.

$$\begin{aligned} sI_1(s) + 50I_2(s) &= \frac{60}{s} \\ -200I_1(s) + (s+200)I_2(s) &= 0. \end{aligned}$$

Solving this system for $I_1(s)$ and $I_2(s)$, we get

$$\begin{aligned} I_1(s) &= \frac{60s + 12,000}{s(s+100)^2} = \frac{6/5}{s} - \frac{6/5}{s+100} - \frac{60}{(s+100)^2} \\ I_2(s) &= \frac{12,000}{s(s+100)^2} = \frac{6/5}{s} - \frac{6/5}{s+100} - \frac{120}{(s+100)^2}. \end{aligned}$$

Hence

$$\begin{aligned} i_1(t) &= \frac{6}{5} - \frac{6}{5}e^{-100t} - 60te^{-100t} \\ i_2(t) &= \frac{6}{5} - \frac{6}{5}e^{-100t} - 120te^{-100t} \end{aligned}$$

Example 4.6.3 (Double pendulum)**.**

$$\begin{aligned} (m_1 + m_2)\ell_1^2\theta_1'' + m_2\ell_1\ell_2\theta_2'' + (m_1 + m_2)\ell_1 g\theta_1 = 0 \\ m_2\ell_2^2\theta_2'' + m_2\ell_1\ell_2\theta_1'' + m_2\ell_2 g\theta_2 = 0. \end{aligned}$$

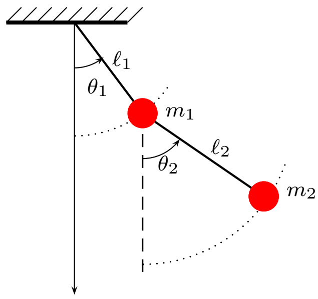


Figure 4.10: double-pendulum